Quasi-Invariants of the Thermophysical Model

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Abstract

The techniques of Lie group analysis can be used to determine absolute invariant functions which serve as classifier functions in object recognition problems. Previously, the Lie groups were found for the conservation equation describing the energy exchange at the surface of an object viewed with an infrared camera. The result was that only trivial absolute invariants exist for the thermophysical model. Consequently, two new topics are presented: First, many model approximations are analyzed based on a thermocouple data set. Second, a formal definition is given for quasi-invariance and selected results are shown for a particular type a quasi-invariant, called a dominant-subspace invariant. More extensive background and results are available in an extended version of this paper.

1 Introduction

Lie group analysis will determine if there exists a non-trivial function Φ which assumes a constant value on the set of all roots of an equation $f(\overline{z}) = 0$. The form of the equation remains constant regardless of which particular object we are measuring (viewing), but some of the coefficients in this equation may (and generally will) change depending upon the object being viewed, as for example when $f(\overline{z}) = 0$ expresses a conservation equation. As a result, the set of roots will differ depending upon the object being viewed. Correspondingly the constant value $\Phi(\overline{z})$ will assume a different value depending upon the object being viewed, thus permitting the use of Φ as a classifier function.

Quasi-invariants are a generalization of absolute invariants. Non-trivial absolute invariants are generally rare. Because the restrictions required for absolute invariance are relaxed, quasi-invariants are more common. These types of functions could be just as useful as absolute invariants in practice. A constructive algorithm is developed to find a particular type of quasi-invariant called a Dominant-Subspace Invariant (DSI). Experimental results validate the approach.

In section 2, several model approximation are presented and evaluated. In section 3, a figure shows the range of thermophysical properties found for typical building materials. Section 4 defines quasi-invariance and presents an algorithm for finding a particular type of quasi-invariant. Finally, experimental data is used to confirm the potential usefulness of these new functions.

2 Approximations to the Thermophysical Model

Many different model variations have been proposed [Incropera and DeWitt, 1981] and used to simplify the conservation equation,

$$\begin{aligned} f \equiv & [W_s \alpha_s \cos\theta + W_l \alpha_l A_{\text{sky}}] \\ & - [\epsilon \sigma T_s^4] - [-h(T_\infty - T_s)] \\ & - [-k \frac{\partial T_s}{\partial z}] - [C_T \frac{\partial T_s}{\partial t}] \end{aligned} \quad (1)$$

Common approximations to the conservation equation include linearizing the radiation term (T_s^4), removing the storage term (thus considering the conservation of energy for a unit surface instead of a control volume), and using finite differences to approximate the partial derivatives. Finally, W_l may be approximated by T_∞^4, or (more appropriately) by T_{sky}^4. The cost of each approximation is quantified in order for the usefulness of the approximations to be more fully understood.

2.1 Approximating W_l

The long-wave insolation term is strongly related to the apparent temperature of the sky. In the absence of a pyranometer to measure this component, a common approximation is to estimate T_{sky}, and use the Stefan-Boltzmann Law such that

$$W_l \approx \epsilon_W \sigma T_{sky}^4.$$

A simpler and more common approximation is to use T_∞

$$W_l \approx \epsilon_W \sigma T_\infty^4.$$

If $\epsilon_W = 1$, this is an idealized blackbody approximation. Figure 2 shows that with the correct selection of ϵ_W (idealized graybody), this is a

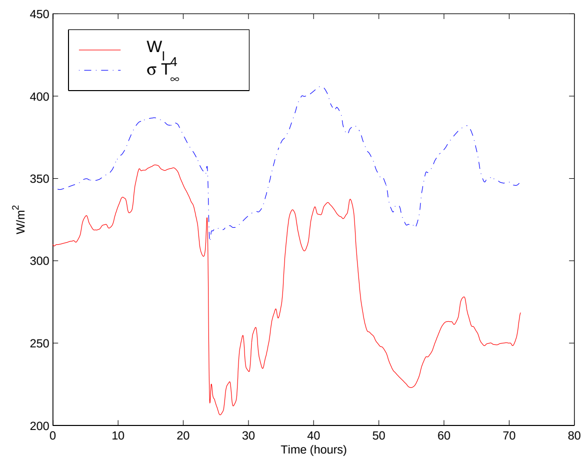

Figure 1: Approximating long-wave insolation (W_l) by an idealized blackbody ($\epsilon_W = 1$) emitter at temperature T_∞. The RMS error is $80\frac{W}{m^2}$ which corresponds to a 28% error.

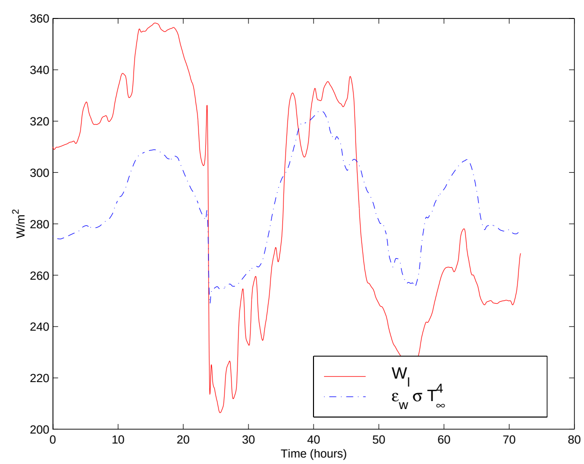

Figure 2: Approximating long-wave insolation (W_l) by an idealized graybody emitter at temperature T_∞. The RMS error is $34\frac{W}{m^2}$ for $\epsilon_W = 0.79$ which corresponds to a 12% error.

fair approximation. The shape of the approximation is close to that of W_l. However, the error is large (in absolute terms) because the mean and variance of the 2 curves do not match. Examining the 3 days of data suggests that the mean and variance should be approximated for no more than a 24 hour period. Such an approximation would be more accurate, but only marginally more complicated than the standard blackbody approximation.

2.2 Net Radiosity Approximation

Another common approximation involves both the long-wave insolation and the convection. Assuming the atmosphere is a blackbody radiator (as above), and that the atmosphere completely surrounds the material, then

$$\begin{aligned} W_l \alpha_l A_{\text{sky}} - \epsilon \sigma T_s^4(t) &\approx \epsilon W_l - \epsilon \sigma T_s^4(t) \\ &\approx \epsilon \sigma (T_\infty^4 - T_s^4(t)) \end{aligned}$$

Note that this is a blackbody approximation because $\epsilon_W = 1$. The ϵ on the right-hand side of the equation comes from $\alpha_l = \epsilon$, not the graybody assumption. The fourth order term is factored to obtain

$$\begin{aligned} &\sigma(T_\infty^4 - T_s^4(t)) \\ &\quad = \sigma(T_\infty + T_s(t))(T_\infty^2 + T_s^2(t))(T_\infty - T_s(t)) \\ &\quad = h_r(T_\infty - T_s(t)) \end{aligned}$$

where h_r is *strongly* dependent on the temperature.

It is common to ignore the strong dependence of h_r on the temperature. For a fixed ambient temperature (say $T_\infty = 15^o$C), and a typical variation in scene temperature (-10^oC to 40^oC) the error due to the linearity assumption is on the order of 10% (see Figure 4). This approximation could be dramatically improved if a linear function of the temperature was used (in place of a constant). However, for the current approximation, the error associated with assuming h_r is constant is negligible compared to the error associated with the blackbody assumption!

This observation also implies that the gray level, L_s, of the LWIR image is related to the surface temperature, T_s, of the object's surface by a linear relationship,

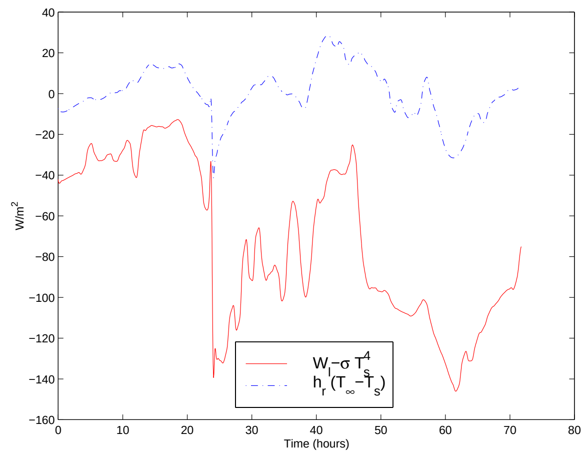

Figure 3: The net radiosity using the blackbody approximation. The error due to the blackbody approximation is unacceptably high. Better approximation methods are suggested in the text.

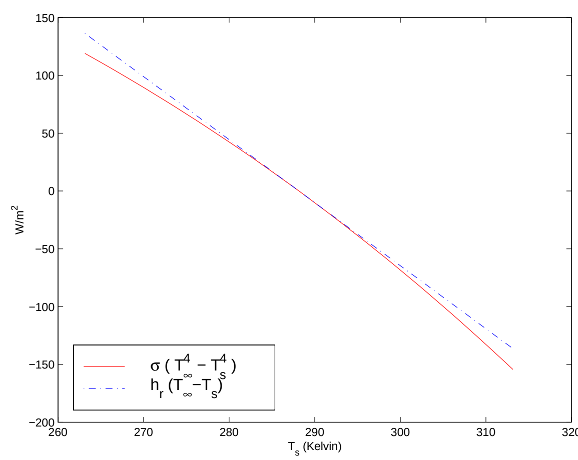

Figure 4: Approximating the net radiosity as a linear function. For $T_\infty = 15^o$C, and $T_s \in [-10^o\text{C}, 40^o\text{C}]$, the error due to this approximation is $\leq 10\%$.

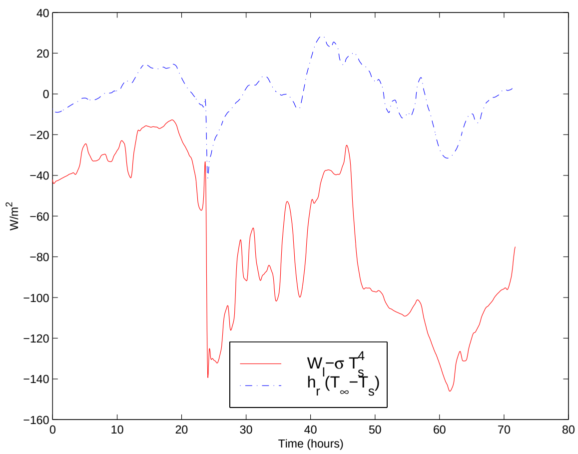

Figure 5: The net radiosity using the blackbody approximation and a constant h_r. The error due to assuming h_r is constant is negligible compared to the error from the blackbody approximation.

$$L_s = \frac{1}{\beta}(T_s + \mu). \qquad (2)$$

where β and μ are related to the IR camera imaging parameters, gain and offset. This approximation is used to create algebraic invariant features that do not require radiometric calibration.

2.3 Thermal Storage Approximation

The conservation statement may be written in terms of a unit surface area instead of a unit volume. The result of considering the surface heat flux is that the thermal storage term is removed from the original equation. This approximation is appropriate for materials with a small thermal capacitance (Figure 6).

2.4 Constant Convection Coefficient

The convection coefficient, h, is commonly assumed to be constant for short time periods (on the order of 24 hours). This approximation is shown to have an error of $\approx 5\frac{W}{m^2}$.

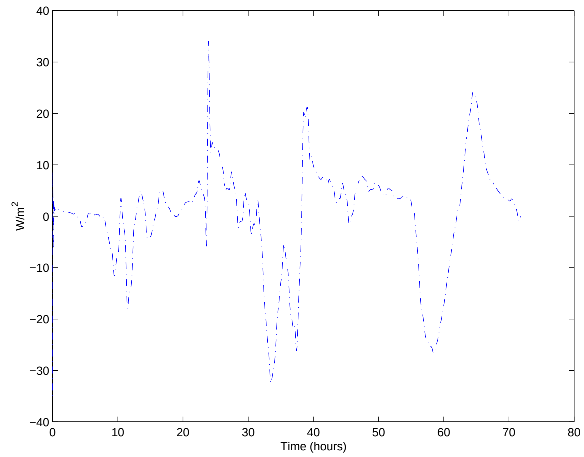

Figure 6: The thermal storage term of concrete over 72 hours. Materials with a low thermal capacitance can be modeled by a heat flux. For concrete (which has an average thermal capacitance), the error due to dropping the storage term is on the order of $11\frac{W}{m^2}$.

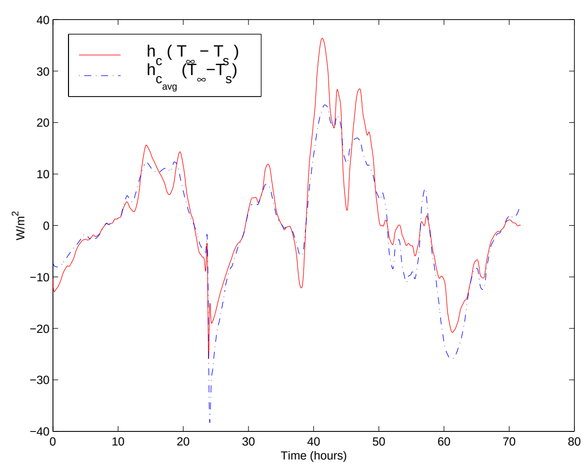

Figure 7: Approximating the convection coefficient as a constant causes an error on the order of $5\frac{W}{m^2}$ in the convection term of the conservation equation.

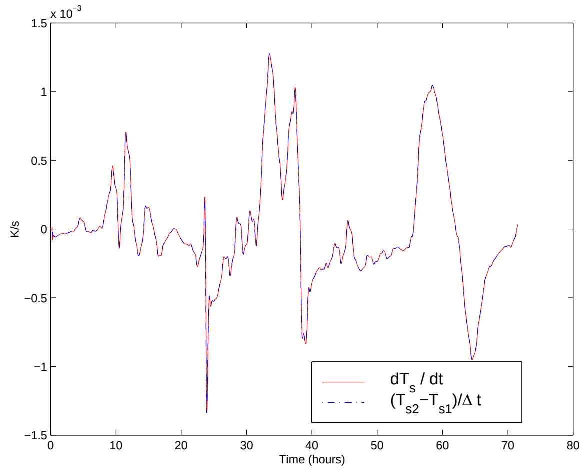

Figure 8: Approximating $\frac{dT_s}{dt} \approx \frac{T_{s_2}-T_{s_1}}{\Delta t}$ introduces very little error for this data set, $0.5\frac{W}{m^2}$ ($\Delta t = 2.5$ minutes).

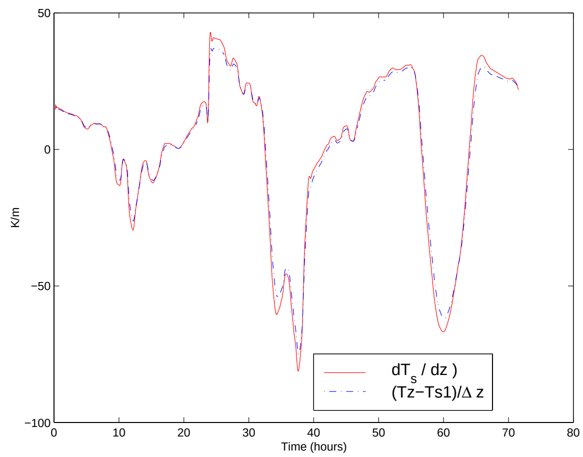

Figure 9: Approximating $\frac{dT_s}{dz} \approx \frac{T_z-T_{s_1}}{\Delta z}$ introduces errors on the order of $10\frac{W}{m^2}$ for $\Delta z = 2.5cm$.

2.5 Approximating the Partial Derivatives

Temperatures are generally smooth and slow-changing. In the absence of analytical equations, derivatives may be approximated by fitting polynomials and calculating the derivatives, or by taking a finite difference. As the time increment is decreased, these methods converge. Many books discuss approximations of derivatives, so we will simply show the difference in using splines or finite-differences as an approximation.

Each of these approximations has a useful purpose. However, the cost with respect to model accuracy must be understood in order to evaluate whether the benefit is commensurate. To help form the proper perspective, we have found that it is not uncommon for radiometric calibration to introduce errors in the surface temperatures on the order of 5 K which could easily produce errors in the conservation equation on the order of $10\frac{W}{m^2}$.

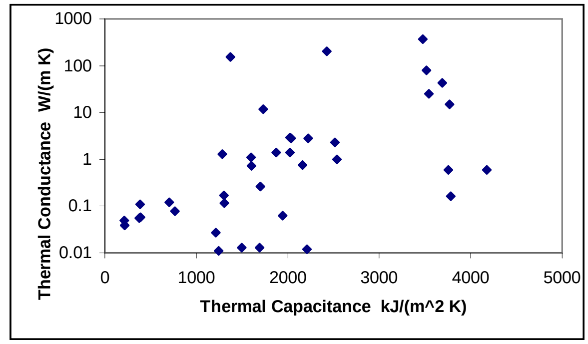

Figure 10: Plotting C_T vs. k reveals that for these 38 construction type materials, some information can be extracted. First, the thermal capacitance increases with the logarithm of the thermal conductance (in general). Second, very few materials have a thermal capacitance around $3000\frac{kJ}{m^3K}$.

3 Groupings of Thermophysical parameters

Classifiers must handle the problem of differentiating between known and unknown classes. Only by "knowing" all the possible classes can results be guaranteed universally. Practically, if common materials have similar bulk material properties, then similar clusters could be used to gain confidence that desirable features will remain useful when used against previously unknown (but similar) classes. Figure 10 shows a scatter plot of thermal conductance versus thermal capacitance for 38 construction type materials drawn from [Brown and Macro, 1958] and various other heat transfer texts. A thorough data collection should be representative of the range and clusters seen in the plot.

Approximation	RMS error	%error	$\frac{W}{m^2}$ error
blackbody	80	28%	80
graybody	34	12%	34
net radiosity	80	96%	80
h_r constant	8	10%	8
net radiosity, h_r constant	80	96%	80
heat flux	-	-	11
constant convection coefficient	-	38%	5
$\frac{T_{s_2}-T_{s_1}}{\Delta t}$	13×10^{-6}	3%	0.5
$\frac{T_z-T_{s_1}}{\Delta z}$	3	10%	10

Table 1: Summary of the error resulting from common approximations for the thermophysical model.

4 Quasi-Invariants

Non-trivial absolute invariants are generally rare. Binford [Binford and Levitt, 1993] defined functions that are "almost always" invariant or "slowly changing" as quasi-invariant functions. Because the restrictions required for absolute invariance are relaxed, quasi-invariants are more common. These types of functions could be just as useful as absolute invariants in practice. The following theoretical concept of quasi-invariance is a mathematically rigorous generalization of Binford's "slowly changing" concept.

A primary goal in applying quasi-invariance theory is to identify the domain in which these features are invariant or nearly invariant (corresponding to the notion of "almost always" invariant). These conditions may result from relationships between some of the variables which are too complex to model. For example, physical constraints often exist which provide bounds on the variables. Lie group analysis with inequalities is not possible. As a consequence an empirical approach is necessary.

4.1 Theoretical Concept

Definition 4.1.1 *A function $\widetilde{\Phi} \in \hom(M_f, \Re)$ is a (δ, ξ) quasi-invariant if for each ${}_\varepsilon\varphi \in S_{M_f}$ the function*

$$\begin{aligned} \widetilde{\Phi} \circ {}_\bullet\varphi(\overline{x}) \quad &: \quad \Re \to \Re \\ &: \quad \varepsilon \mapsto \widetilde{\Phi} \circ {}_\varepsilon\varphi(\overline{x}) \end{aligned}$$

satisfies the conditions of continuity at 0 with the pair (δ, ξ),

$$|\varepsilon - 0| \leq \delta \Rightarrow |\widetilde{\Phi}({}_\varepsilon\varphi(\overline{x})) - \widetilde{\Phi}({}_0\varphi(\overline{x}))| \leq \xi$$

Since the elements, ${}_\varepsilon\varphi(\overline{x})$, of the symmetry group S_{M_f} satisfy ${}_{\alpha+\beta}\varphi = {}_\alpha\varphi \circ {}_\beta\varphi$ it follows that continuity at zero implies uniform continuity,

$$|(\alpha + \beta) - \beta| = |\alpha - 0| \leq \delta$$

implies

$$\begin{aligned} &|\widetilde{\Phi}({}_{\alpha+\beta}\varphi(\overline{x})) - \widetilde{\Phi}({}_\beta\varphi(\overline{x}))| \\ &= |\widetilde{\Phi}({}_\alpha\varphi(\overline{y})) - \widetilde{\Phi}({}_0\varphi(\overline{y}))| \leq \xi \end{aligned}$$

where

$$\overline{y} \equiv {}_\beta\varphi(\overline{x}) = {}_0\varphi(\overline{y})$$

Obviously if $\widetilde{\Phi}$ is continuous then $\widetilde{\Phi} \circ {}_\varepsilon\varphi(\overline{x})$ is continuous, so given an arbitrary ξ there exists a δ satisfying the condition of continuity. We are concerned with the converse problem: Given a δ, find an ξ such that the condition of continuity is satisfied at 0 with the pair (δ, ξ). (Given a specific application, one seeks to determine the value ξ as small as possible such that the required condition is satisfied.)

One would expect an absolute invariant Φ to satisfy the conditions for being a quasi-invariant.

Theorem 4.1.2 *If Φ is an absolute invariant, then for any positive pair (δ, ξ), Φ is a (δ, ξ) quasi-invariant.*

Proof. This is immediate since for an absolute function Φ

$$\Phi({}_{\varepsilon}\varphi(\overline{x})) = \Phi(\overline{x}) \qquad \forall_{\varepsilon}\varphi(\overline{x}) \in S_{M_f} \quad \forall \varepsilon$$

so for any $\delta \geq 0$ and any $\xi \geq 0$

$$\begin{array}{rcl} |\varepsilon - 0| & \leq & \delta \\ & \Downarrow & \\ |\Phi({}_{\varepsilon}\varphi(\overline{x})) - \Phi({}_{0}\varphi(\overline{x}))| & = & |\Phi({}_{\varepsilon}\varphi(\overline{x})) - \Phi(\overline{x})| \\ & = & 0 \leq \xi \end{array}$$

■

The motivation for the definition of a (δ, ξ) quasi-invariant with respect to object recognition is the following: Given a quasi-invariant function, $\widetilde{\Phi}_j$ for each class j, and a measurement $\overline{x}_k$, then $\widetilde{\Phi}_j(\overline{x}_k)$ varies slowly if $j = k$, thereby satisfying the (δ_j, ξ_j) conditions. If $j \neq k$, then $\widetilde{\Phi}_j(\overline{x}_k)$ will not vary slowly, and therefore the (δ_j, ξ_j) condition will *not* be satisfied. The problem of separating classes is directly related to the variance of $\widetilde{\Phi}_j(\overline{x}_k), j \neq k$. This immediately implies that a time sequence of data will generally be necessary.

Another possible method for use of quasi-invariant functions is to develop one function, $\widetilde{\Phi}$, for classes of the same form but different parameters. A typical range of $\widetilde{\Phi}$ can be determined for each class. Then for a given measurement $\overline{x}_j$, $\widetilde{\Phi}(\overline{x}_j)$ serves as an indexing function into class j. Simulated or empirical data is necessary to determine which classes are separable (by different values of this function). This second formulation would allow identification with a single data point, however it is unlikely that the different classes can be described by the same quasi-invariant form with different parameters.

4.2 An Algorithm for Determining Quasi-invariants

No guarantee is given that useful quasi-invariants exist (trivial ones always exist). However, the following construction algorithm is one method to seek out non-trivial (and hopefully useful) quasi-invariants. We call these dominant-subspace invariants (DSI's). Since a single method for finding useful quasi-invariants does not exist, any technique that produces a quasi-invariant that satisfies the definition is valid.

The tangent vectors (infinitesimal generators) are a vector field set which forms a basis for the killing fields. The tangent vectors $\{\overline{\eta}_i\}_{i=1,\ldots,m}$ determine the transformation groups (we will use the convention $m = n - 1$ throughout the remainder of the paper). Absolute invariant functions are found by solving the characteristic equations associated with these tangent vectors. A necessary and sufficient condition for a function Φ to be invariant under the symmetry group determined by the generators is

$$\overline{\eta}_i(\Phi) = 0 \qquad i = 1, \ldots, m$$

Absolute invariants are rare, and therefore by relaxing the requirements for absolute invariance, the chance of finding a useful classifier is increased.

A (δ, ξ) quasi-invariant function $\widetilde{\Phi}$ is a function which satisfies

$$\begin{array}{rcl} |\varepsilon - 0| & \leq & \delta \\ & \Downarrow & \\ |\Phi({}_{\varepsilon}\varphi(\overline{x})) - \Phi({}_{0}\varphi(\overline{x}))| & = & |\Phi({}_{\varepsilon}\varphi(\overline{x})) - \Phi(\overline{x})| \\ & = & 0 \leq \xi \end{array}$$

For a small δ, the goal is to find as tight a bound ξ as possible, thereby making $\widetilde{\Phi}$ nearly constant. This condition is equivalent to finding a function $\widetilde{\Phi}$ such that the derivative is small in magnitude

$$\left\| \frac{d\widetilde{\Phi}({}_{\varepsilon}\varphi(\overline{x}))}{d\varepsilon} \right\| \approx 0 \tag{3}$$

Theoretically, if this quantity is nonzero, then by speeding up (scaling) the curve, ${}_{\varepsilon}\varphi$, this quantity can be made arbitrarily large. Therefore, the necessary restriction is that the quantity be empirically small. Since

$$\frac{d\widetilde{\Phi}({}_{\varepsilon}\varphi(\overline{x}))}{d\varepsilon} = \sum_{j=1}^{n} \left.\frac{\partial \widetilde{\Phi}}{\partial x_j}\right|_{{}_{0}\varphi(\overline{x})} \frac{d_{\varepsilon}\varphi_{x_j}(\overline{x})}{d\varepsilon} = \frac{d_{\varepsilon}\varphi(\overline{x})}{d\varepsilon}\widetilde{\Phi} \tag{4}$$

it is equivalent to seek $\widetilde{\Phi}$ such that

$$\left\| \frac{d_{\varepsilon}\varphi(\overline{x})}{d\varepsilon}\widetilde{\Phi} \right\| \approx 0 \tag{5}$$

Solving the characteristic equations associated with the tangent vectors $\{\overline{\eta}_i\}_{i=1,\ldots,m}$ yields the invariant Φ. Solving the characteristic equations associated with some subset of $\{\overline{\eta}_i\}_{i=1,\ldots,m}$, say without loss of generality $\{\overline{\eta}_h\}_{h=1,\ldots,m-1}$ yields a function $\widetilde{\Phi}$. The subset defines a subspace, and the function satisfies

$$\overline{\eta}_h\widetilde{\Phi} = 0 \qquad h = 1,\ldots,m-1 \tag{6}$$

while

$$\overline{\eta}_m\widetilde{\Phi} \neq 0 \tag{7}$$

Thus solving these characteristic equations gives

$$\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon}\widetilde{\Phi} = 0$$

where $\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} = \sum_{h=1}^{m-1} g_h\,\overline{\eta}_h$. Coupling this with the desired property that $\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\widetilde{\Phi}$ be small in magnitude gives the equations

$$\begin{aligned}\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon}\widetilde{\Phi} &= 0\\ \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\widetilde{\Phi} &\approx 0\end{aligned}$$

Subtracting the two, and using linearity of tangent vectors gives

$$\left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\widetilde{\Phi} \approx 0 \tag{8}$$

But

$$\begin{aligned}&\left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\widetilde{\Phi}\\ &= \left\langle \nabla\widetilde{\Phi}, \left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\right\rangle\end{aligned}$$

and by the Cauchy-Schwartz inequality,

$$\begin{aligned}&\left\|\left\langle \nabla\widetilde{\Phi}, \left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\right\rangle\right\|\\ &\qquad\leq \left\|\nabla\widetilde{\Phi}\right\| \left\|\left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\right\|\end{aligned}$$

$\nabla\widetilde{\Phi}\Big|_{{}_\alpha\varphi(\overline{x})} = \left\{\frac{\partial\widetilde{\Phi}}{\partial x_1}\Big|_{{}_\alpha\varphi(\overline{x})},\ldots,\frac{\partial\widetilde{\Phi}}{\partial x_n}\Big|_{{}_\alpha\varphi(\overline{x})}\right\}$ is a bounded vector since $\forall\alpha\in[0,\delta]$ the restriction of $\nabla\widetilde{\Phi}\circ{}_\alpha\varphi(\overline{x})$ to $[0,\delta]$ gives a continuous image of a compact set- hence $\nabla\widetilde{\Phi}\Big|_{{}_\alpha\varphi(\overline{x})}$ is compact, therefore bounded.

Now, say $\left\|\nabla\widetilde{\Phi}\right\| \leq K$. Then to make $\left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\widetilde{\Phi}$ as small as possible, minimize

$$\left\|\left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\right\| \tag{9}$$

The tightest bound is achieved if $\left\|\nabla\widetilde{\Phi}\right\|$ is not only bounded, but constant.

This is one method for minimizing the inner product. Other methods may form a tighter bound on equation (8). This new minimization replaces the last original constraint for an absolute invariant

$$\overline{\eta}_m\widetilde{\Phi} = 0$$

Consider the differential equation characterizing the transformation group of ${}_\varepsilon\varphi$

$$\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon} = \left(\sum_{i=1}^{m} g_i\overline{\eta}_i\right)({}_\varepsilon\varphi(\overline{x})) \qquad {}_0\varphi(\overline{x}) = \overline{x} \tag{10}$$

The curves can be expressed as functions relative to the moving basis $\overline{\overline{\eta}}$ (we use 'moving basis' because 'moving frame' generally implies an orthonormal basis). By curve fitting experimental data, the vector fields $\frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}$ can be determined. The vector fields, $\overline{\overline{\eta}}$, define a local coordinate system for the surface derivatives, and are derived analytically by Lie group analysis. Therefore, the scalar coefficients $\overline{g}\in C^1(\Re^m)$ can be determined from the other terms. By decomposing the moving basis into principal components, a subspace of the tangent space determined by the infinitesimal generators may be discovered. Such a subspace could be the result of "overlooking" some physical constraint that is not accounted for by the single equation modeling the problem – the conservation of energy equation.

Rewriting

$$\begin{aligned}&\left\|\left(\frac{d\widetilde{{}_\varepsilon\varphi}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon\varphi(\overline{x})}{d\varepsilon}\right)\right\|\\ &\quad= \left\|\left(\sum_{i=1}^{m-1} g_{s_i}\overline{\eta}_{s_i} - \sum_{i=1}^{m} g_{s_i}\overline{\eta}_{s_i}\right)({}_\varepsilon\varphi(\overline{x}))\right\|\\ &\quad= \left\|\left(-g_{s_m}\overline{\eta}_{s_m}\right)({}_\varepsilon\varphi(\overline{x}))\right\|\end{aligned}$$

equation (7) indicates that $\overline{\eta}_{s_m}\widetilde{\Phi}\neq 0$. The test for a subspace is performed by comparing

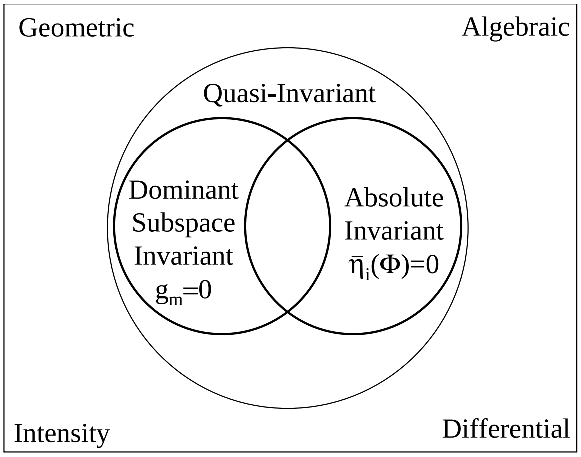

Figure 11: Diagram of the relation between different types of invariants.

$$\left\| \frac{d_\varepsilon \widetilde{\varphi}_{x_j}(\overline{x})}{d\varepsilon} - \frac{d_\varepsilon \varphi_{x_j}(\overline{x})}{d\varepsilon} \right\| \ll \left\| \frac{d_\varepsilon \varphi_{x_j}(\overline{x})}{d\varepsilon} \right\|.$$

In the limiting case, $g_{s_m} = 0$, a (non-trivial) true subspace invariant exists! The distinction from an absolute invariant is in the definition. An absolute invariant function is defined by its vanishing inner product with a basis element $\overline{\eta}_i$, but a DSI is defined by a vanishing coefficient, g_{s_m}. The result, $\Phi = constant$, is identical. Note, we can *not* conclude that absolute invariants do not exist under the case that DSI's do not exist. In other words, if (non-trivial) DSI's do not exist, then (non-trivial) absolute invariants may still exist. See Figure 11. This argument may be relaxed to $\|g_{s_m}\| \approx 0$ for $\|\overline{\eta}_{s_m}\|$ sufficiently small.

4.2.1 A Constructive Algorithm for Calculating Dominant Subspace Invariants

1. Create unitless variables by dividing each variable in the conservation equation, f, by its RMS value, and multiplying its corresponding coefficient by the same RMS value. This new conservation equation is f_u.

2. Analytically determine a set of vector fields, $\overline{\overline{\eta}}$, which form a basis for the killing fields of f_u.

3. Compute the coefficients of the commutator table (discussed below). Find a new basis, $\overline{\overline{\eta}}_o$, such that all the coefficients are 0.

4. Find the principal components of $\overline{\overline{\eta}}_o$ to determine a new basis, $\overline{\overline{\eta}}_s$, for the killing fields such that equation (9) is minimized.

5. Solve the set of characteristic equations corresponding to $\overline{\eta}_{s_h} \widetilde{\Phi} = 0$ for $h = 1, \ldots, m-1$ to find the general form of $\widetilde{\Phi}$. This part of the calculation of the DSI proceeds exactly as it did in the determination of (absolute) invariants except that the final infinitesimal generator has not been explicitly satisfied.

6. Further experimentation is necessary to determine if the DSI is useful.

5 Results

Applying this algorithm to a thermocouple data set, we found the following dominant-subspace invariant

$$\widetilde{\Phi}\left[S_a\, x_S + S_b\, x_L + S_c\, x_H + S_d\, x_A + S_e\, x_1 - x_Z\right] \tag{11}$$

where the $S_\bullet$ are constants that are dependent upon the particular material.

In Figure 12, the dark dotted line represents $\widetilde{\Phi}$ when the observed material matches the hypothesized material, concrete, during a 72 hours test. The remaining lines illustrate that $\widetilde{\Phi}$ changes its characteristic if the hypothesis does not match the observed material. Concrete cannot be separated from painted concrete. Concrete (plain and painted) clearly yields the best DSI in terms of relative stability, separation, and generalization.

6 Summary

The techniques of Lie group analysis provide a powerful tool for determining absolute invariant functions which can serve as classifier functions for object recognition problems. Previously, we applied this analysis to the thermophysical model and proved that there are only trivial absolute invariant functions.

	S_a	S_b	S_c	S_d	S_e
Asphalt	-0.002	-0.051	0.001	0.591	0.095
Painted Asphalt	-0.002	-0.042	0.000	0.511	0.083
Concrete	-0.001	-0.006	-0.000	0.005	0.947
Painted Concrete	-0.000	-0.005	-0.000	-0.004	0.949
Clay	-0.002	-0.014	-0.000	0.046	0.916
Gravel	-0.002	-0.014	-0.000	0.066	0.866
Grass	-0.002	-0.030	0.000	0.327	0.403

Table 2: Coefficients for the 'principal' basis and dominant-subspace invariant for each material.

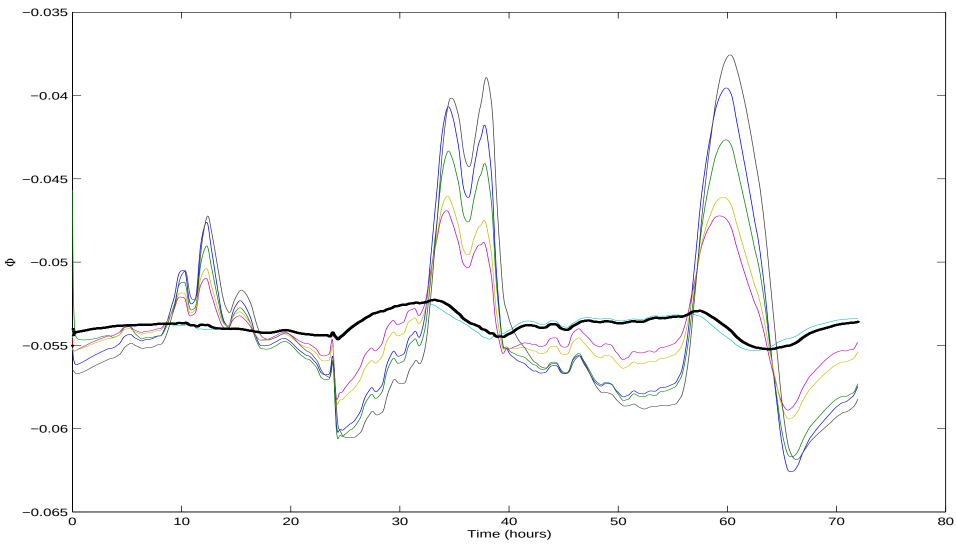

Figure 12: 72 hours of results for the concrete hypothesis.

We present quantitative errors for several model approximations. We also presented a definition for quasi-invariance, and an algorithm for finding a particular type of quasi-invariant called a dominant-subspace invariant. We found a dominant-subspace invariant for the thermophysical model and illustrated the ideal result as applied to concrete. Further details and results are available by contacting the authors.

References

[Binford and Levitt, 1993] Thomas O. Binford and Tod S. Levitt. Quasi-invariants: Theory and exploitation. In O. Firschein, editor, *DARPA Image Understanding Workshop Proceedings*, pages 819–830, Washington DC, 1993. Morgan Kauffman.

[Brown and Macro, 1958] A.I. Brown and S.M. Macro. *Introduction to Heat Transfer*. McGraw-Hill, New York, 1958.

[Incropera and DeWitt, 1981] F.P. Incropera and D.P. DeWitt. *Fundamentals of Heat Transfer*. John Wiley and Sons, New York, 1981.